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Reordering and Reflecting Factors for Simulation Studies With Exploratory Factor Analysis

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This article describes the REREFAC T R package, which provides a postrotation algorithm that reorders or reflects factors for each replication of a simulation study with exploratory factor analysis (EFA). The purpose of REREFAC T is to provide a general algorithm written in freely available software, R, dedicated to addressing the possibility that a nonuniform order or sign pattern of the factors could be observed across replications. The algorithm implemented in REREFAC T proceeds in 4 steps. Step 1 determines the total number of equivalent forms, I , of the vector of factors, $\boldsymbol{\eta}$. Step 2 indexes, $i = 1, 2 \dots I$, each equivalent form of $\boldsymbol{\eta}$ (i.e., $\boldsymbol{\eta}^i$) via a unique permutation matrix, \mathbf{P} (i.e., \mathbf{P}^i). Step 3 determines which $\boldsymbol{\eta}^i$ each replication follows. Step 4 uses the appropriate \mathbf{P}^i to reorder or re-sign parameter estimates within each replication so that all replications uniformly follow the order and sign pattern defined by the population values. Results from two simulation studies provided evidence for the efficacy of the REREFAC T to identify and remediate equivalent forms of $\boldsymbol{\eta}$ in models with EFA only (i.e., Example 1) and in fuller parameterizations of exploratory structural equation modeling (i.e., Example 2). How to use REREFAC T is briefly demonstrated prior to the Discussion section by providing annotations for key commands and condensed output using a subset of simulated data from Example 1.

Keywords: exploratory structural equation modeling, factor order, Monte Carlo, negative factor, permutation matrices, simulation

Asparouhov and Muthén (2009) reminded readers of a well-known issue regarding the order and sign pattern within the vector of factors, $\boldsymbol{\eta}$, in models with exploratory factor analysis (EFA): “In ESEM [exploratory structural equation modeling], as well as EFA, the order of all factors is interchangeable and each factor is interchangeable with its negative” (p. 436). The potential importance of the issue in practice, however, might depend on the number of data sets being analyzed. When analyzing a single data set, such as a real data analysis with EFA, Asparouhov and Muthén noted that “these indeterminacies [order and sign pattern within $\boldsymbol{\eta}$] are typically unimportant” (p. 436). When analyzing multiple data sets, such as a simulation study with EFA, Asparouhov and Muthén cautioned, “However, they [the indeterminacies] are important in simulation studies where accumulations across different

replications are done to evaluate MSE [mean square error], parameter estimates bias, and confidence interval coverage” (p. 436). The expression *indeterminacies within $\boldsymbol{\eta}$* is used in this article to concisely refer to the arbitrary nature of the order and sign pattern within $\boldsymbol{\eta}$ in models with EFA.

Four considerations for evaluating the potential importance of the indeterminacies within $\boldsymbol{\eta}$ when analyzing a single data set guided the development of the algorithm implemented in REREFAC T for addressing this issue in simulation studies with EFA when analyzing multiple data sets. One consideration was determining the existence of equivalent forms of $\boldsymbol{\eta}$ with regard to model–data misfit. A second consideration was that equivalent forms of $\boldsymbol{\eta}$ can produce nonequivalent outcomes. A third consideration was the ease with which an equivalent form of $\boldsymbol{\eta}$ could be manually identified, particularly when the interpretation of the results is guided by an a priori theory. A fourth consideration was the ease with which an equivalent, but undesirable, form of $\boldsymbol{\eta}$ can be manually remedied. Each of these four considerations for evaluating the potential importance of the indeterminacies

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within $\boldsymbol{\eta}$ is elaborated on in subsequent sections with an EFA-only example (i.e., Example 1) while introducing the algorithm implemented in, and providing results from, REREFAC. Implications for simulation studies with fuller parameterizations of ESEM are briefly covered in Example 2. The final section prior to the Discussion section briefly demonstrates how to use the REREFAC R package.

EQUIVALENT FORMS OF $\boldsymbol{\eta}$

The expression *equivalent forms of $\boldsymbol{\eta}$* is used in this article to describe models with EFA with equivalent model–data misfit where only the order or sign pattern within $\boldsymbol{\eta}$, and related reorganization (e.g., the order of η_1 and η_2 is changed) or re-signing (e.g., η_1 is replaced with its negative, $-\eta_1$) of model parameters may differ. The reorganization or re-signing of model parameters that could be expected given the indeterminacies within $\boldsymbol{\eta}$ can be demonstrated through a formal presentation of the model. The single population ESEM for continuous endogenous variables can be written with two equations, where Equation 1 is the measurement model and Equation 2 is the latent variable model (Bollen, 1989). Reduced versions of both equations are presented initially to focus on implications for model parameters due to indeterminacies within $\boldsymbol{\eta}$ in models with EFA (without means) only:

$$\mathbf{y}_{pxl} = \boldsymbol{\Lambda}_{pxm} \boldsymbol{\eta}_{mxl} + \boldsymbol{\varepsilon}_{pxl} \quad (1)$$

$$\boldsymbol{\eta}_{mxl} = \boldsymbol{\zeta}_{mxl}, \text{ where} \quad (2)$$

p is the number of observed endogenous variables, m is the number of latent variables, \mathbf{y} is the vector of observed endogenous variables, $\boldsymbol{\Lambda}$ is the matrix of (\mathbf{y} on $\boldsymbol{\eta}$) pattern coefficients, $\boldsymbol{\eta}$ is the vector of latent variables, $\boldsymbol{\varepsilon}$ is the vector of residuals for \mathbf{y} , $\boldsymbol{\Theta}$ is the pxp covariance matrix for $\boldsymbol{\varepsilon}$, $\boldsymbol{\zeta}$ is the vector of residuals for $\boldsymbol{\eta}$, and $\boldsymbol{\Psi}$ is the $m \times m$ covariance matrix for $\boldsymbol{\zeta}$.

As can be seen in Equation 1 any order or sign pattern changes within $\boldsymbol{\eta}$ would have implications for order or sign pattern changes of the m columns of $\boldsymbol{\Lambda}$ only (i.e., $\boldsymbol{\Theta}$ would be unaffected). As can be inferred from Equation 2 (and the definition of terms after it) any order or sign pattern changes within $\boldsymbol{\eta}$ would have implications for order or sign pattern changes within the m columns and rows of the symmetric $\boldsymbol{\Psi}$.

Model–data misfit is indexed in this article by the value of the discrepancy function under maximum likelihood (ML) estimation, F_{ML} .¹ For the example introduced in the next paragraph $F_{ML} = \ln|\hat{\boldsymbol{\Sigma}}| + \text{tr}(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}) - \ln|\boldsymbol{\Sigma}| - p$, where $\hat{\boldsymbol{\Sigma}}$ is the

covariance matrix implied by the model, given by $\boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$, and $\boldsymbol{\Sigma}$ is the population data. Thus, a necessary condition for equivalent model–data misfit for equivalent forms of $\boldsymbol{\eta}$ is an equal F_{ML} value, which is achieved by a $\hat{\boldsymbol{\Sigma}}$ that is identical to $\boldsymbol{\Sigma}$ when $\boldsymbol{\Sigma}$ is available. In a sample, the observed covariance matrix, \mathbf{S} , is substituted for $\boldsymbol{\Sigma}$. Thus, a necessary condition for equivalent model–data misfit for equivalent forms of $\boldsymbol{\eta}$ is an equal \hat{F}_{ML} value when an \mathbf{S} is available.

Example 1

An example that assumes no model error is provided to demonstrate equivalent forms of $\boldsymbol{\eta}$ for a three-factor EFA model with 10 continuous indicators. Parameter values were selected so that indicator communality, h_s , ranged from .20 to .80 with $\bar{h} = .53$. These values were consistent with EFA models in practice, per results from Henson and Roberts (2006) where the average total variance explained by the extracted factors $\approx 52\%$ and ranged from 17% to 87%. Parameter values within $\boldsymbol{\Lambda}$ were selected to reflect practice in the sense that high values were not identical and low values varied around, but were unequal to, zero (MacCallum, Widaman, Zhang, & Hong, 1999; Marsh et al., 2009).² Parameter values within $\boldsymbol{\Psi}$ were selected to reflect practice in the sense that a priori substantive theory often suggests the presence of nonzero values in at least some of the off-diagonal elements (e.g., Fabrigar, Wegener, MacCallum, & Strahan, 1999). Table 1 provides $\boldsymbol{\Sigma}$ and an \mathbf{S} . Sample data were drawn from a multivariate normal distribution ($\mathbf{0}, \boldsymbol{\Sigma}$) with $N = 300$ to reflect a level of sampling error in \mathbf{S} that is common in practice. Henson and Roberts (2006) reported that the distribution of N for EFA models in practice is positively skewed ($Mdn = 267, M = 436$).

Three equivalent forms of $\boldsymbol{\eta}$ were selected for demonstration purposes. Equation 3 through Equation 5 provide values for $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, F_{ML} , and \hat{F}_{ML} for three equivalent forms of $\boldsymbol{\eta}$.³ For each equivalent form, $\boldsymbol{\Theta}$ is a diagonal matrix with the following rounded values along the diagonal: 0.38, 0.84, 0.21, 0.38, 0.44, 1.18, 0.44, 0.40, 0.58, and 0.35. From this point forward a superscript following an array, F_{ML} or \hat{F}_{ML} indexes a particular equivalent form. Form 1 is given in Equation 3 and is equal to the case of no order or sign change for any individual factor, η_k , within $\boldsymbol{\eta}$ (i.e., $\boldsymbol{\eta}^{1'} = [\eta_1^1 \eta_2^1 \eta_3^1]$):

² Values within $\boldsymbol{\Lambda}$ were very similar to values in the 10-variable, three-factor, wide condition of MacCallum et al. (1999) and were accessed at http://www.unc.edu/~rcm/article_material.htm.

³ For each equivalent form, F_{ML} and \hat{F}_{ML} were found by substituting the relevant data (i.e., $\boldsymbol{\Sigma}$ and \mathbf{S} , respectively) and $\hat{\boldsymbol{\Sigma}}$ into the discrepancy function provided previously. The parameter values provided in Equation 3 through Equation 5 were used to find $\hat{\boldsymbol{\Sigma}}$.

¹ ML estimation was chosen because it is commonly employed in simulation studies (e.g., Bandalos & Leite, 2013; Hoogland & Boomsma, 1998) including those for which the potential use of REREFAC would be relevant. The general implications of indeterminacies within $\boldsymbol{\eta}$ in models with EFA, however, would seem to be relevant for other estimation methods as well.

TABLE 1
Population and Sample Covariance and Standard Deviation Values for the Observed Variables From Example 1

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
y_1	—	<i>0.44</i>	<i>-0.18</i>	<i>0.07</i>	<i>0.23</i>	<i>0.14</i>	<i>0.43</i>	<i>-0.04</i>	<i>-0.05</i>	<i>0.08</i>
y_2	<i>0.36</i>	—	<i>-0.06</i>	<i>0.09</i>	<i>0.38</i>	<i>0.15</i>	<i>0.52</i>	<i>0.14</i>	<i>0.23</i>	<i>0.31</i>
y_3	<i>-0.15</i>	<i>0.05</i>	—	<i>0.41</i>	<i>0.29</i>	<i>0.33</i>	<i>0.15</i>	<i>0.78</i>	<i>0.29</i>	<i>0.27</i>
y_4	<i>0.13</i>	<i>0.13</i>	<i>0.47</i>	—	<i>0.12</i>	<i>0.07</i>	<i>0.09</i>	<i>0.39</i>	<i>0.12</i>	<i>0.08</i>
y_5	<i>0.18</i>	<i>0.39</i>	<i>0.34</i>	<i>0.20</i>	—	<i>0.44</i>	<i>0.60</i>	<i>0.48</i>	<i>0.41</i>	<i>0.56</i>
y_6	<i>0.09</i>	<i>0.23</i>	<i>0.37</i>	<i>0.23</i>	<i>0.41</i>	—	<i>0.37</i>	<i>0.45</i>	<i>0.24</i>	<i>0.38</i>
y_7	<i>0.29</i>	<i>0.44</i>	<i>0.22</i>	<i>0.17</i>	<i>0.64</i>	<i>0.38</i>	—	<i>0.40</i>	<i>0.43</i>	<i>0.59</i>
y_8	<i>0.00</i>	<i>0.23</i>	<i>0.84</i>	<i>0.50</i>	<i>0.57</i>	<i>0.49</i>	<i>0.46</i>	—	<i>0.43</i>	<i>0.46</i>
y_9	<i>0.03</i>	<i>0.24</i>	<i>0.30</i>	<i>0.13</i>	<i>0.49</i>	<i>0.31</i>	<i>0.45</i>	<i>0.45</i>	—	<i>0.42</i>
y_{10}	<i>0.05</i>	<i>0.29</i>	<i>0.31</i>	<i>0.13</i>	<i>0.58</i>	<i>0.35</i>	<i>0.53</i>	<i>0.49</i>	<i>0.45</i>	—
<i>SD</i>	<i>0.99</i>	<i>1.12</i>	<i>1.00</i>	<i>0.81</i>	<i>1.03</i>	<i>1.15</i>	<i>1.08</i>	<i>1.11</i>	<i>0.96</i>	<i>0.94</i>
<i>SD</i>	<i>0.97</i>	<i>1.10</i>	<i>1.02</i>	<i>0.86</i>	<i>1.05</i>	<i>1.22</i>	<i>1.04</i>	<i>1.16</i>	<i>0.98</i>	<i>0.94</i>

Note. Sample values are shown in italics. Values are rounded.

$$\Lambda^1 = \begin{bmatrix} -0.07 & 0.03 & 0.77 \\ 0.35 & -0.03 & 0.42 \\ 0.05 & 0.87 & -0.20 \\ -0.16 & 0.65 & 0.21 \\ 0.77 & 0.03 & 0.10 \\ 0.36 & 0.25 & 0.06 \\ 0.72 & -0.05 & 0.26 \\ 0.33 & 0.77 & -0.06 \\ 0.63 & 0.01 & -0.07 \\ 0.77 & -0.05 & -0.08 \end{bmatrix}, \Psi^1$$

$$= \begin{bmatrix} 1.00 & .50 & .25 \\ .50 & 1.00 & .00 \\ .25 & .00 & 1.00 \end{bmatrix}, F_{ML}^1 = 0, \hat{F}_{ML}^1$$

$$= .18. \tag{3}$$

From this point forward Form 1 is considered the desired form from which order changes, sign changes, or both can occur in an equivalent form of η . Form 2 is given in Equation 4 and is equal to the case of order change for η_1^1 and η_2^1 and no sign change for any η_k^1 (i.e., $\eta^{2'} = [\eta_2^1 \ \eta_1^1 \ \eta_3^1]$):

$$\Lambda^2 = \begin{bmatrix} 0.03 & -0.07 & 0.77 \\ -0.03 & 0.35 & 0.42 \\ 0.87 & 0.05 & -0.20 \\ 0.65 & -0.16 & 0.21 \\ 0.03 & 0.77 & 0.10 \\ 0.25 & 0.36 & 0.06 \\ -0.05 & 0.72 & 0.26 \\ 0.77 & 0.33 & -0.06 \\ 0.01 & 0.63 & -0.07 \\ -0.05 & 0.77 & -0.08 \end{bmatrix}, \Psi^2$$

$$= \begin{bmatrix} 1.00 & .50 & .00 \\ .50 & 1.00 & .25 \\ .00 & .25 & 1.00 \end{bmatrix}, F_{ML}^2 = 0, \hat{F}_{ML}^2$$

$$= .18. \tag{4}$$

Form 3 is given in Equation 5 and is equal to the case of no order change for any η_k^1 and a sign change for η_1^1 (i.e., $\eta^{3'} = [-\eta_1^1 \ \eta_2^1 \ \eta_3^1]$):

$$\Lambda^3 = \begin{bmatrix} 0.07 & 0.03 & 0.77 \\ -0.35 & -0.03 & 0.42 \\ -0.05 & 0.87 & -0.20 \\ 0.16 & 0.65 & 0.21 \\ -0.77 & 0.03 & 0.10 \\ -0.36 & 0.25 & 0.06 \\ -0.72 & -0.05 & 0.26 \\ -0.33 & 0.77 & -0.06 \\ -0.63 & 0.01 & -0.07 \\ -0.77 & -0.05 & -0.08 \end{bmatrix}, \Psi^3$$

$$= \begin{bmatrix} 1.00 & -.50 & -.25 \\ -.50 & 1.00 & .00 \\ -.25 & .00 & 1.00 \end{bmatrix}, F_{ML}^3 = 0, \hat{F}_{ML}^3$$

$$= .18. \tag{5}$$

To summarize, Form 1 through Form 3 are equivalent because model-data misfit is equivalent for a given data set and only order or sign pattern within η and related reorganization or re-signing of model parameters differs. For example, the order change within $\eta^{2'} = [\eta_2^1 \ \eta_1^1 \ \eta_3^1]$ was linked with the reordering column 1 (i.e., λ_{p1}^1) and column 2 (i.e., λ_{p2}^1) within Λ^2 , and, the reordering of ψ_{13}^1 and ψ_{23}^1 within Ψ^2 . Similarly, the sign change within $\eta^{3'} = [-\eta_1^1 \ \eta_2^1 \ \eta_3^1]$ was linked with the re-signing of λ_{p1}^1 (i.e., $-\lambda_{p1}^1$) within Λ^3 , and, the re-signing of both ψ_{12}^1 (i.e., $-\psi_{12}^1$) and ψ_{13}^1 (i.e., $-\psi_{13}^1$) within Ψ^3 . In summary, and irrespective of the number of data sets, the existence of equivalent forms of η due to the indeterminacies within η can be viewed as unimportant with regard to model-data fit for a given data set.

REREFRACT ALGORITHM

Step 1 of the algorithm is to determine I : the total number of equivalent forms of η that can result from the arbitrary nature of both the order and sign of the m rows of η . For any η , I is equal to the product of $m! * \sum_{s=0}^m C(m, s)$, where C is combination. The expression $m!$ accounts for all possible reorderings of the m rows. The expression $\sum_{s=0}^m C(m, s)$ accounts for all possible sign changes of m rows.⁴

Step 2 of the algorithm is to index within I via a unique, orthogonal permutation matrix for each equivalent form of η . Let i serve to index each unique equivalent form of η and range from $i = 1, 2, \dots, I$ and be denoted by η^i . Let P^i be a permutation matrix that is a row permutation of an $m \times m$ identity matrix, where every column and every row contain one nonzero entry that is 1 (i.e., accounting for all possible reorderings of the m rows η) or -1 (i.e., accounting for all possible sign changes of m rows η), to satisfy the following equation: $P^i \eta = \eta^i$.⁵ The indexing for P^i is identical to the indexing of η^i and ranges from $i = 1, 2, \dots, I$. The population (or, equivalently, the desired) order and sign pattern of η is defined as $P^I \eta = \eta^I$ by setting $P^1 = I$.

REREFRACT RESULTS

In Example 1 there are three factors (i.e., $\eta_{3 \times 1}$), and thus, $I = 48$ because $m! * \sum_{s=0}^m C(m, s) = 6 * 8$. Table 2 provides each of the 48 P^i and corresponding η^i . As can be seen in Table 2, the algorithm proceeds by determining all unique sign assignments for a given ordering within η before proceeding to a different ordering within η . For this reason the indexing in Table 2 differs from the indexing used for the three equivalent forms of η provided in the previous section except for Form 1 (i.e., Form 2 and Form 3 from the previous section are η^{41} and η^2 , respectively, in Table 2). Replacing η with $P^i \eta$ in Equation 1 and Equation 2 leads to the following expressions that describe the reorganization or re-signing of model parameters that could be expected given the indeterminacies within η :

$$\Lambda^i = \Lambda P^{i'} \tag{6}$$

$$\Psi^i = P^i \Psi P^{i'} \tag{7}$$

⁴ An equivalent expression for I when $m \geq 2$ is $m! * 2^m$ (e.g., Ichikawa & Konishi, 1995).

⁵ As a group the P^i can be described as signed permutation matrices (a subgroup of generalized permutation matrices) instead of permutation matrices because nonzero entries are ± 1 . A more detailed document on the indexing of P^i for each equivalent form of η is available on request.

NONEQUIVALENT OUTCOMES

Equivalent forms of η can produce nonequivalent outcomes. Values for some outcomes commonly of interest in simulation studies with EFA (e.g., parameter estimate bias, mean square error [MSE], and congruence coefficient) often are not equivalent across equivalent forms of η . For the i th parameter, ω_i , over the number of replications, n_r , parameter estimate bias is given by:

$$bias(\hat{\omega}_i) = \sum_{j=1}^{n_r} \frac{(\hat{\omega}_{ij} - \omega_i)}{n_r} \tag{8}$$

Mean square error is given by:

$$MSE(\hat{\omega}_i) = \sum_{j=1}^{n_r} \frac{(\hat{\omega}_{ij} - \omega_i)^2}{n_r} \tag{9}$$

For both bias and MSE a value of zero indicates the absence of bias (and variability of the parameter estimate in the case of MSE). Bias and MSE values within an array (e.g., Λ) often are averaged across at least some elements within an array for each condition of a simulation study with EFA (e.g., MacCallum, Widaman, Preacher, & Hong, 2001; Sass & Schmitt, 2010). Congruence coefficient for each η_k with regard to the pattern coefficients is given by:

$$\phi_k = \frac{\sum_{i=1}^p \hat{\lambda}_{ik} \lambda_{ik}}{\sqrt{\left(\sum_{i=1}^p \hat{\lambda}_{ik}^2\right) \left(\sum_{i=1}^p \lambda_{ik}^2\right)}} \tag{10}$$

An average congruence coefficient across factors, K , is given by:

$$K = \frac{\sum_{k=1}^m \phi_k}{m} \tag{11}$$

A K value of 1.00 indicates maximum similarity between $\hat{\Lambda}$ and Λ (e.g., Burt, 1948). K values often are an outcome of interest in simulation studies with EFA (e.g., de Winter, Dodou, & Wieringa, 2009; Myers, Ahn, & Jin, 2013). In real data analysis with a single data set, the fact that equivalent forms of η can produce nonequivalent outcomes often is unimportant because bias, MSE , and K are rarely of interest in these studies where the population parameter values are typically unknown.

Example 1

Outcome values for equivalent forms of η were derived without estimation by treating parameter values in Equation 3 as the population values (i.e., $\Lambda^1 = \Lambda$ and $\Psi^1 = \Psi$), and by

TABLE 2

Indexing (i) Equivalent Forms of the Vector of Factors, η [η_1 ; η_2 ; η_3], From Example 1 Via Permutation Matrices, \mathbf{P}^i , Where $\mathbf{P}^i\eta = \eta^i$

\mathbf{P}^i	η^i	F_{ML}	F_{ML}
$\mathbf{P}^1 = [1,0,0; 0,1,0; 0,0,1]$	$\eta^1 = [\eta_1^1; \eta_2^1; \eta_3^1]$.00	.18
$\mathbf{P}^2 = [-1,0,0; 0,1,0; 0,0,1]$	$\eta^2 = [-\eta_1^1; \eta_2^1; \eta_3^1]$.00	.18
$\mathbf{P}^3 = [1,0,0; 0,-1,0; 0,0,1]$	$\eta^3 = [\eta_1^1; -\eta_2^1; \eta_3^1]$.00	.18
$\mathbf{P}^4 = [-1,0,0; 0,-1,0; 0,0,1]$	$\eta^4 = [-\eta_1^1; -\eta_2^1; \eta_3^1]$.00	.18
$\mathbf{P}^5 = [1,0,0; 0,1,0; 0,0,-1]$	$\eta^5 = [\eta_1^1; \eta_2^1; -\eta_3^1]$.00	.18
$\mathbf{P}^6 = [-1,0,0; 0,1,0; 0,0,-1]$	$\eta^6 = [-\eta_1^1; \eta_2^1; -\eta_3^1]$.00	.18
$\mathbf{P}^7 = [1,0,0; 0,-1,0; 0,0,-1]$	$\eta^7 = [\eta_1^1; -\eta_2^1; -\eta_3^1]$.00	.18
$\mathbf{P}^8 = [-1,0,0; 0,-1,0; 0,0,-1]$	$\eta^8 = [-\eta_1^1; -\eta_2^1; -\eta_3^1]$.00	.18
$\mathbf{P}^9 = [1,0,0; 0,0,1; 0,1,0]$	$\eta^9 = [\eta_1^1; \eta_3^1; \eta_2^1]$.00	.18
$\mathbf{P}^{10} = [-1,0,0; 0,0,1; 0,1,0]$	$\eta^{10} = [-\eta_1^1; \eta_3^1; \eta_2^1]$.00	.18
$\mathbf{P}^{11} = [1,0,0; 0,0,-1; 0,1,0]$	$\eta^{11} = [\eta_1^1; -\eta_3^1; \eta_2^1]$.00	.18
$\mathbf{P}^{12} = [-1,0,0; 0,0,-1; 0,1,0]$	$\eta^{12} = [-\eta_1^1; -\eta_3^1; \eta_2^1]$.00	.18
$\mathbf{P}^{13} = [1,0,0; 0,0,1; 0,-1,0]$	$\eta^{13} = [\eta_1^1; \eta_3^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{14} = [-1,0,0; 0,0,1; 0,-1,0]$	$\eta^{14} = [-\eta_1^1; \eta_3^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{15} = [1,0,0; 0,0,-1; 0,-1,0]$	$\eta^{15} = [\eta_1^1; -\eta_3^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{16} = [-1,0,0; 0,0,-1; 0,-1,0]$	$\eta^{16} = [-\eta_1^1; -\eta_3^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{17} = [0,0,1; 1,0,0; 0,1,0]$	$\eta^{17} = [\eta_3^1; \eta_1^1; \eta_2^1]$.00	.18
$\mathbf{P}^{18} = [0,0,-1; 1,0,0; 0,1,0]$	$\eta^{18} = [-\eta_3^1; \eta_1^1; \eta_2^1]$.00	.18
$\mathbf{P}^{19} = [0,0,1; -1,0,0; 0,1,0]$	$\eta^{19} = [\eta_3^1; -\eta_1^1; \eta_2^1]$.00	.18
$\mathbf{P}^{20} = [0,0,-1; -1,0,0; 0,1,0]$	$\eta^{20} = [-\eta_3^1; -\eta_1^1; \eta_2^1]$.00	.18
$\mathbf{P}^{21} = [0,0,1; 1,0,0; 0,-1,0]$	$\eta^{21} = [\eta_3^1; \eta_1^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{22} = [0,0,-1; 1,0,0; 0,-1,0]$	$\eta^{22} = [-\eta_3^1; \eta_1^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{23} = [0,0,1; -1,0,0; 0,-1,0]$	$\eta^{23} = [\eta_3^1; -\eta_1^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{24} = [0,0,-1; -1,0,0; 0,-1,0]$	$\eta^{24} = [-\eta_3^1; -\eta_1^1; -\eta_2^1]$.00	.18
$\mathbf{P}^{25} = [0,0,1; 0,1,0; 1,0,0]$	$\eta^{25} = [\eta_3^1; \eta_2^1; \eta_1^1]$.00	.18
$\mathbf{P}^{26} = [0,0,-1; 0,1,0; 1,0,0]$	$\eta^{26} = [-\eta_3^1; \eta_2^1; \eta_1^1]$.00	.18
$\mathbf{P}^{27} = [0,0,1; 0,-1,0; 1,0,0]$	$\eta^{27} = [\eta_3^1; -\eta_2^1; \eta_1^1]$.00	.18
$\mathbf{P}^{28} = [0,0,-1; 0,-1,0; 1,0,0]$	$\eta^{28} = [-\eta_3^1; -\eta_2^1; \eta_1^1]$.00	.18
$\mathbf{P}^{29} = [0,0,1; 0,1,0; -1,0,0]$	$\eta^{29} = [\eta_3^1; \eta_2^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{30} = [0,0,-1; 0,1,0; -1,0,0]$	$\eta^{30} = [-\eta_3^1; \eta_2^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{31} = [0,0,1; 0,-1,0; -1,0,0]$	$\eta^{31} = [\eta_3^1; -\eta_2^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{32} = [0,0,-1; 0,-1,0; -1,0,0]$	$\eta^{32} = [-\eta_3^1; -\eta_2^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{33} = [0,1,0; 0,0,1; 1,0,0]$	$\eta^{33} = [\eta_2^1; \eta_3^1; \eta_1^1]$.00	.18
$\mathbf{P}^{34} = [0,-1,0; 0,0,1; 1,0,0]$	$\eta^{34} = [-\eta_2^1; \eta_3^1; \eta_1^1]$.00	.18
$\mathbf{P}^{35} = [0,1,0; 0,0,-1; 1,0,0]$	$\eta^{35} = [\eta_2^1; -\eta_3^1; \eta_1^1]$.00	.18
$\mathbf{P}^{36} = [0,-1,0; 0,0,-1; 1,0,0]$	$\eta^{36} = [-\eta_2^1; -\eta_3^1; \eta_1^1]$.00	.18
$\mathbf{P}^{37} = [0,1,0; 0,0,1; -1,0,0]$	$\eta^{37} = [\eta_2^1; \eta_3^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{38} = [0,-1,0; 0,0,1; -1,0,0]$	$\eta^{38} = [-\eta_2^1; \eta_3^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{39} = [0,1,0; 0,0,-1; -1,0,0]$	$\eta^{39} = [\eta_2^1; -\eta_3^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{40} = [0,-1,0; 0,0,-1; -1,0,0]$	$\eta^{40} = [-\eta_2^1; -\eta_3^1; -\eta_1^1]$.00	.18
$\mathbf{P}^{41} = [0,1,0; 1,0,0; 0,0,1]$	$\eta^{41} = [\eta_2^1; \eta_1^1; \eta_3^1]$.00	.18
$\mathbf{P}^{42} = [0,-1,0; 1,0,0; 0,0,1]$	$\eta^{42} = [-\eta_2^1; \eta_1^1; \eta_3^1]$.00	.18
$\mathbf{P}^{43} = [0,1,0; -1,0,0; 0,0,1]$	$\eta^{43} = [\eta_2^1; -\eta_1^1; \eta_3^1]$.00	.18
$\mathbf{P}^{44} = [0,-1,0; -1,0,0; 0,0,1]$	$\eta^{44} = [-\eta_2^1; -\eta_1^1; \eta_3^1]$.00	.18
$\mathbf{P}^{45} = [0,1,0; 1,0,0; 0,0,-1]$	$\eta^{45} = [\eta_2^1; \eta_1^1; -\eta_3^1]$.00	.18
$\mathbf{P}^{46} = [0,-1,0; 1,0,0; 0,0,-1]$	$\eta^{46} = [-\eta_2^1; \eta_1^1; -\eta_3^1]$.00	.18
$\mathbf{P}^{47} = [0,1,0; -1,0,0; 0,0,-1]$	$\eta^{47} = [\eta_2^1; -\eta_1^1; -\eta_3^1]$.00	.18
$\mathbf{P}^{48} = [0,-1,0; -1,0,0; 0,0,-1]$	$\eta^{48} = [-\eta_2^1; -\eta_1^1; -\eta_3^1]$.00	.18

Note. [] = an array; , = separates elements within a row; ; = separates rows;

F_{ML} = discrepancy function value under maximum likelihood (ML) estimation with population data (Σ) provided in Table 1; F_{ML} = discrepancy function value under ML estimation with sample data (\mathbf{S}) provided in Table 1.

using Equation 6 and Equation 7 (\mathbf{P}^i provided in Table 2), to find values within each of the 48 Λ^i and Ψ^i , which were then treated as parameter estimates. For example, treating the values in Equation 3, found by $\Lambda^1 = \Lambda\mathbf{P}^{1'}$ and $\Psi^1 = \mathbf{P}^1\Psi\mathbf{P}^{1'}$, as parameter estimates provides the following values for Form 1 (or equivalently, η^1 in Table 3):

$$\begin{aligned} bias(\hat{\Lambda}) &= \mathbf{0}, bias(\hat{\Psi}) = \mathbf{0}, \phi_1 = 1.00, \\ \phi_2 &= 1.00, \phi_3 = 1.00. \end{aligned} \tag{12}$$

The *MSE* for each parameter estimate can be found by squaring the relevant value within $bias(\hat{\Lambda})$ or $bias(\hat{\Psi})$. Treating the values in Equation 4, found by $\Lambda^{41} = \Lambda\mathbf{P}^{41'}$ and $\Psi^{41} = \mathbf{P}^{41}\Psi\mathbf{P}^{41'}$, as parameter estimates provides the following values for Form 2 (or equivalently, η^{41} in Table 3):

$$\begin{aligned} bias(\hat{\Lambda}) &= \begin{bmatrix} 0.10 & -0.10 & 0.00 \\ -0.38 & 0.38 & 0.00 \\ 0.82 & -0.82 & 0.00 \\ 0.81 & -0.81 & 0.00 \\ -0.74 & 0.74 & 0.00 \\ -0.11 & 0.11 & 0.00 \\ -0.77 & 0.77 & 0.00 \\ 0.44 & -0.44 & 0.00 \\ -0.62 & 0.62 & 0.00 \\ -0.82 & 0.82 & 0.00 \end{bmatrix}, bias(\hat{\Psi}) \\ &= \begin{bmatrix} 0.00 & & \\ 0.00 & 0.00 & \\ -0.25 & 0.25 & 0.00 \end{bmatrix}, \phi_1 = .11, \phi_2 \\ &= .11, \phi_3 = 1.00. \end{aligned} \tag{13}$$

Note that in both Equation 12 and Equation 13 the average value within $bias(\hat{\Lambda})$ and $bias(\hat{\Psi})$ equals 0.00 but in the case of Equation 13 this is due to the canceling of nonzero values and not to the absence of bias within either array. Note, too, that in both Equation 12 and Equation 13 ϕ_3 equals 1.00 and both ϕ_1 and ϕ_2 are considerably less than 1.00 in Equation 13 (but equal 1.00 in Equation 12), which suggests that K might be a more reliable indicator, as compared to individual ϕ_k , of similarity between $\hat{\Lambda}$ and Λ . Average *MSE* with regard to $\hat{\Lambda}$ equals 0.26 in Equation 13 and in Equation 12 the value equals 0.00. Average *MSE* with regard to $\hat{\Psi}$ equals 0.02 in Equation 13 and in Equation 12 the value equals 0.00.

Table 3 provides outcome values for all 48 equivalent forms of η with population data. Average bias with regard to $\hat{\Lambda}$ equals 0.00 each time there is a unique order and no sign change for any η_k^i within η (i.e., $\eta^1, \eta^9, \eta^{17}, \eta^{25}, \eta^{33}, \eta^{41}$). Average bias with regard to $\hat{\Psi}$ equals 0.00 when there is sign change for every η_k^i within η (i.e., $\eta^8, \eta^{16}, \eta^{24}, \eta^{32}, \eta^{40}$) in addition to

TABLE 3
Some Outcome Values for Equivalent Forms of the Vector of Factors, η [η_1 ; η_2 ; η_3], From Example 1 With Population Data

η^i	Average Bias		Average MSE		Congruence
	$\hat{\Lambda}$	$\hat{\Psi}$	$\hat{\Lambda}$	$\hat{\Psi}$	K (ϕ_1, ϕ_2, ϕ_3)
$\eta^1 = [\eta_1^1; \eta_2^1; \eta_3^1]$	0.00	0.00	0.00	0.00	1.00 (1.00,1.00,1.00)
$\eta^2 = [-\eta_1^1; \eta_2^1; \eta_3^1]$	-0.25	-0.25	0.33	0.21	.33 (-1.00,1.00,1.00)
$\eta^3 = [\eta_1^1; -\eta_2^1; \eta_3^1]$	-0.17	-0.17	0.25	0.17	.33 (1.00,-1.00,1.00)
$\eta^4 = [-\eta_1^1; -\eta_2^1; \eta_3^1]$	-0.42	-0.08	0.58	0.04	-.33 (-1.00,-1.00,1.00)
$\eta^5 = [\eta_1^1; \eta_2^1; -\eta_3^1]$	-0.09	-0.08	0.13	0.04	.33 (1.00,1.00,-1.00)
$\eta^6 = [-\eta_1^1; \eta_2^1; -\eta_3^1]$	-0.34	-0.17	0.46	0.17	-.33 (-1.00,1.00,-1.00)
$\eta^7 = [\eta_1^1; -\eta_2^1; -\eta_3^1]$	-0.26	-0.25	0.37	0.21	-.33 (1.00,-1.00,-1.00)
$\eta^8 = [-\eta_1^1; -\eta_2^1; -\eta_3^1]$	-0.51	0.00	0.70	0.00	-1.00 (-1.00,-1.00,-1.00)
$\eta^9 = [\eta_1^1; \eta_3^1; \eta_2^1]$	0.00	0.00	0.19	0.02	.30 (1.00,-.05,-.05)
$\eta^{10} = [-\eta_1^1; \eta_3^1; \eta_2^1]$	-0.25	-0.25	0.53	0.19	-.37 (-1.00,-.05,-.05)
$\eta^{11} = [\eta_1^1; -\eta_3^1; \eta_2^1]$	-0.09	-0.08	0.19	0.10	.33 (1.00,.05,-.05)
$\eta^{12} = [-\eta_1^1; -\eta_3^1; \eta_2^1]$	-0.34	-0.17	0.52	0.10	-.33 (-1.00,.05,-.05)
$\eta^{13} = [\eta_1^1; \eta_3^1; -\eta_2^1]$	-0.17	-0.17	0.19	0.10	.33 (1.00,-.05,.05)
$\eta^{14} = [-\eta_1^1; \eta_3^1; -\eta_2^1]$	-0.42	-0.08	0.52	0.10	-.33 (-1.00,-.05,.05)
$\eta^{15} = [\eta_1^1; -\eta_3^1; -\eta_2^1]$	-0.26	-0.25	0.18	0.19	.37 (1.00,.05,.05)
$\eta^{16} = [-\eta_1^1; -\eta_3^1; -\eta_2^1]$	-0.51	0.00	0.51	0.02	-.30 (-1.00,.05,.05)
$\eta^{17} = [\eta_3^1; \eta_1^1; \eta_2^1]$	0.00	0.00	0.33	0.06	.06 (.14,.11,-.05)
$\eta^{18} = [-\eta_3^1; \eta_1^1; \eta_2^1]$	-0.09	-0.08	0.36	0.15	-.03 (-.14,.11,-.05)
$\eta^{19} = [\eta_3^1; -\eta_1^1; \eta_2^1]$	-0.25	-0.25	0.36	0.15	-.01 (.14,-.11,-.05)
$\eta^{20} = [-\eta_3^1; -\eta_1^1; \eta_2^1]$	-0.34	-0.17	0.39	0.06	-.10 (-.14,-.11,-.05)
$\eta^{21} = [\eta_3^1; \eta_1^1; -\eta_2^1]$	-0.17	-0.17	0.32	0.06	.10 (.14,.11,.05)
$\eta^{22} = [-\eta_3^1; \eta_1^1; -\eta_2^1]$	-0.26	-0.25	0.35	0.15	.01 (-.14,.11,.05)
$\eta^{23} = [\eta_3^1; -\eta_1^1; -\eta_2^1]$	-0.42	-0.08	0.35	0.15	.03 (.14,-.11,.05)
$\eta^{24} = [-\eta_3^1; -\eta_1^1; -\eta_2^1]$	-0.51	0.00	0.38	0.06	-.06 (-.14,-.11,.05)
$\eta^{25} = [\eta_3^1; \eta_2^1; \eta_1^1]$	0.00	0.00	0.20	0.08	.42 (.14,1.00,.14)
$\eta^{26} = [-\eta_3^1; \eta_2^1; \eta_1^1]$	-0.09	-0.08	0.23	0.13	.33 (-.14,1.00,.14)
$\eta^{27} = [\eta_3^1; -\eta_2^1; \eta_1^1]$	-0.17	-0.17	0.45	0.08	-.24 (.14,-1.00,.14)
$\eta^{28} = [-\eta_3^1; -\eta_2^1; \eta_1^1]$	-0.26	-0.25	0.48	0.13	-.33 (-.14,-1.00,.14)
$\eta^{29} = [\eta_3^1; \eta_2^1; -\eta_1^1]$	-0.25	-0.25	0.23	0.13	.33 (.14,1.00,-.14)
$\eta^{30} = [-\eta_3^1; \eta_2^1; -\eta_1^1]$	-0.34	-0.17	0.26	0.08	.24 (-.14,1.00,-.14)
$\eta^{31} = [\eta_3^1; -\eta_2^1; -\eta_1^1]$	-0.42	-0.08	0.48	0.13	-.33 (.14,-1.00,-.14)
$\eta^{32} = [-\eta_3^1; -\eta_2^1; -\eta_1^1]$	-0.51	0.00	0.50	0.08	-.42 (-.14,-1.00,-.14)
$\eta^{33} = [\eta_2^1; \eta_3^1; \eta_1^1]$	0.00	0.00	0.33	0.06	.06 (.11,-.05,.14)
$\eta^{34} = [-\eta_2^1; \eta_3^1; \eta_1^1]$	-0.17	-0.17	0.36	0.15	-.01 (-.11,-.05,.14)
$\eta^{35} = [\eta_2^1; -\eta_3^1; \eta_1^1]$	-0.09	-0.08	0.32	0.06	.10 (.11,.05,.14)
$\eta^{36} = [-\eta_2^1; -\eta_3^1; \eta_1^1]$	-0.26	-0.25	0.35	0.15	.03 (-.11,-.05,.14)
$\eta^{37} = [\eta_2^1; \eta_3^1; -\eta_1^1]$	-0.25	-0.25	0.36	0.15	-.03 (.11,-.05,-.14)
$\eta^{38} = [-\eta_2^1; \eta_3^1; -\eta_1^1]$	-0.42	-0.08	0.39	0.06	-.10 (-.11,-.05,-.14)
$\eta^{39} = [\eta_2^1; -\eta_3^1; -\eta_1^1]$	-0.34	-0.17	0.35	0.15	.01 (.11,.05,-.14)
$\eta^{40} = [-\eta_2^1; -\eta_3^1; -\eta_1^1]$	-0.51	0.00	0.38	0.06	-.06 (-.11,.05,-.14)
$\eta^{41} = [\eta_2^1; \eta_1^1; \eta_3^1]$	0.00	0.00	0.26	0.02	.40 (.11,1.11.00)
$\eta^{42} = [-\eta_2^1; \eta_1^1; \eta_3^1]$	-0.17	-0.17	0.29	0.19	.33 (-.11,.11,1.00)
$\eta^{43} = [\eta_2^1; -\eta_1^1; \eta_3^1]$	-0.25	-0.25	0.29	0.19	.33 (.11,-.11,1.00)
$\eta^{44} = [-\eta_2^1; -\eta_1^1; \eta_3^1]$	-0.42	-0.08	0.32	0.02	.26 (-.11,-.11,1.00)
$\eta^{45} = [\eta_2^1; \eta_1^1; -\eta_3^1]$	-0.09	-0.08	0.39	0.02	-.26 (.11,.11,-1.00)
$\eta^{46} = [-\eta_2^1; \eta_1^1; -\eta_3^1]$	-0.26	-0.25	0.42	0.19	-.33 (-.11,.11,-1.00)
$\eta^{47} = [\eta_2^1; -\eta_1^1; -\eta_3^1]$	-0.34	-0.17	0.42	0.19	-.33 (.11,-.11,-1.00)
$\eta^{48} = [-\eta_2^1; -\eta_1^1; -\eta_3^1]$	-0.51	0.00	0.45	0.02	-.40 (-.11,-.11,-1.00)

Note. The indexing of η^i follows the indexing established in Table 2. MSE = mean square error; [] = an array; ; = separates rows.

those cases when average bias with regard to $\hat{\Lambda}$ equals 0.00. Average MSE with regard to $\hat{\Lambda}$ equals 0.00 for only equivalent form η^1 . Average MSE with regard to $\hat{\Psi}$ equals 0.00 for equivalent forms η^1 and η^8 in the latter case because the

order is correct while the sign change is uniform across η_k^1 : $\eta^{8i} = [-\eta_1^1 \ -\eta_2^1 \ -\eta_3^1]$. Finally, K equals 1.00 for only equivalent form η^1 . In summary, equivalent forms of η often produce nonequivalent outcomes that are commonly of

interest, and therefore this is an important consideration, in simulation studies with EFA.

IDENTIFICATION AND REMEDIATION OF AN EQUIVALENT FORM OF η : SINGLE DATA SET

Identification and remediation of an equivalent, but undesirable, form of η might be easy to accomplish manually when working with a single data set, particularly when the interpretation of the results is guided by an a priori theory. The option of manually changing the sign pattern within arrays of parameter estimates that can be expected to be affected by an undesirable sign pattern within η to improve interpretation for a single data set has been noted by many scholars and for some time: “Any factor with a preponderance of negative salient loadings can always be reversed. One simply multiplies the factor (i.e., its loadings and correlations with other factors) by -1 to reverse the factor’s direction if that is necessary to achieve more positive loadings” (Gorsuch, 1983, p. 181). Similarly, and for a single data set, the irrelevance of the order within arrays of parameter estimates that can be expected to be affected by an undesirable order within η also has been noted by many scholars and for some time: “In general, the order of the factors is not of interest in evaluating the solution” (Thompson, 2004, p. 96). Manual solutions to the observation of an undesirable order within η (e.g., $\eta^{41'} = [\eta_2^1 \ \eta_1^1 \ \eta_3^1]$) include reordering affected arrays of parameter estimates within tables or to reorder the a priori factor names to match the observed order in the output (e.g., the a priori name for η_1^1 is reassigned to Factor 2).

Example 1

A priori theory typically is incomplete in practice (e.g., MacCallum, 2003; Marcoulides & Drezner, 2003). Incomplete a priori theory can be used both to inform a rotation criterion in EFA and to help identify an equivalent form of η postrotation. EFA with target rotation is a way to infuse incomplete a priori measurement theory into the rotation criterion and such an approach has been a focus of some recent simulation studies (e.g., Dolan, Oort, Stoel, & Wicherts, 2009; Reise, Moore, & Maydeu-Olivares, 2011). Target rotation was designed to rotate Λ to a least squares fit to a targeted pattern matrix, \mathbf{B} , without fixing coefficients to particular values (e.g., Browne, 1972, 2001). The continuous complexity function minimized under target rotation can be written as:

$$\sum_{i=1}^p \sum_{k=1}^m a_{ik} (\lambda_{ik} - b_{ik})^2, \text{ where} \tag{14}$$

$a_{ik} = 1$ if λ_{ik} is a target and 0 if λ_{ik} is not a target, and b_{ik} = the targeted value.

The sample data set summarized in Table 1 was fit to a three-factor EFA with oblique target rotation within a

simulation study (i.e., the number of repetitions equaled 1) in *Mplus* 7.31 (Muthén & Muthén, 1998–2012). Parameter values were provided in the model statement to define the alignment criteria to reduce the likelihood of observing an equivalent form of η (Asparouhov & Muthén, 2009). Diagonal entries within Ψ were fixed to 1.00. Two small-valued pattern coefficients in each column were specified as targets.⁶ Suppose that a priori theory suggested a lead indicator for each factor (e.g., Gagné & Hancock, 2006; Millsap, 2001). More specifically, a large positive value (denoted by a + symbol in \mathbf{B}) was expected (but not targeted in the rotation algorithm) for each of the following pattern coefficients: $\lambda_{10,1}, \lambda_{3,2}, \lambda_{1,3}$. The + symbols in \mathbf{B} can be thought of as an incomplete a priori theory that can be used to help to identify an equivalent form of η postrotation. The \mathbf{B} (where a value denotes a target) and the parameter estimates from the rotated solution are provided here:

$$\mathbf{B} = \begin{bmatrix} -0.07 & 0.03 & + \\ ? & ? & ? \\ 0.05 & + & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & 0.06 \\ ? & ? & ? \\ ? & ? & -0.06 \\ ? & 0.01 & ? \\ + & ? & ? \end{bmatrix}, \tag{15}$$

$$\hat{\Lambda} = \begin{bmatrix} -0.07 & 0.03 & -0.99 \\ 0.56 & 0.24 & -0.02 \\ 0.05 & -0.99 & -0.29 \\ -0.13 & -0.59 & -0.45 \\ 0.91 & 0.04 & 0.30 \\ 0.52 & -0.17 & 0.07 \\ 1.03 & 0.24 & 0.27 \\ 0.47 & -0.75 & -0.07 \\ 0.77 & 0.01 & 0.49 \\ 1.04 & 0.15 & 0.59 \end{bmatrix},$$

$$(\hat{\Psi}) = \begin{bmatrix} -0.08 \\ -0.63 & -0.46 \end{bmatrix}.$$

⁶ Imposing m^2 constraints on Λ and Ψ was a necessary condition for rotation identification (e.g., Algina, 1980; Hayashi & Marcoulides, 2006). A set of sufficient conditions for imposing the remaining $m(m - 1)$ constraints included: (a) each column of Λ had $m - 1$ elements specified as zero, and (b) each submatrix Λ_s , where $s = 1, \dots, m$, of Λ composed of the rows of Λ that have fixed zeros in the k th column must have rank $m - 1$. Consistent with Asparouhov and Muthén (2009), small values (in this case, $\leq |.07|$) were counted as approximately zero.

Comparing the parameter estimates in Equation 15 to the parameter values in Equation 3 (i.e., Λ^1, Ψ^1) provides the following outcome values:

$$\begin{aligned} \text{bias}(\hat{\Lambda}) &= \begin{bmatrix} 0.00 & 0.00 & -1.76 \\ 0.21 & 0.27 & -0.44 \\ 0.00 & -1.86 & -0.09 \\ 0.03 & -1.24 & -0.66 \\ 0.14 & 0.01 & 0.20 \\ 0.16 & -0.42 & 0.01 \\ 0.31 & 0.29 & 0.01 \\ 0.14 & -1.52 & -0.01 \\ 0.14 & 0.00 & 0.56 \\ 0.27 & 0.20 & 0.67 \end{bmatrix}, \text{bias}(\hat{\Psi}) \\ &= \begin{bmatrix} -0.58 \\ -0.88 & -0.46 \end{bmatrix}, \phi_1 = 1.00, \phi_2 \\ &= -.97, \phi_3 = -.56. \end{aligned} \quad (16)$$

Values within Equation 15 (e.g., $\hat{\lambda}_{10,1} = 1.04, \hat{\lambda}_{3,2} = -0.99, \hat{\lambda}_{1,3} = -0.99$) suggest, however, that an undesired equivalent form, $\eta^{7'} = [\eta_1^1 \quad -\eta_2^1 \quad -\eta_3^1]$, was observed. Parameter estimates from Equation 15, now indexed with a 7 superscript (consistent with Table 2), can be re-signed manually (e.g., re-sign $\hat{\psi}_{21}, \hat{\psi}_{31}$ and each parameter estimate within $\hat{\lambda}_{p2}$ and $\hat{\lambda}_{p3}$) or by using \mathbf{P}^7 (see Table 2) to follow the desired equivalent form, $\eta^{1'} = [\eta_1^1 \quad \eta_2^1 \quad \eta_3^1]$:

$$\hat{\Lambda}^7 \mathbf{P}^7 = \hat{\Lambda}^1 \quad (17)$$

$$\mathbf{P}^{7'} \hat{\Psi}^7 \mathbf{P}^7 = \hat{\Psi}^1. \quad (18)$$

Comparing the re-signed parameter estimates to the parameter values in Equation 3 provides the following outcome values:

$$\begin{aligned} \text{bias}(\hat{\Lambda}) &= \begin{bmatrix} 0.00 & -0.06 & 0.22 \\ 0.21 & -0.21 & -0.40 \\ 0.00 & 0.12 & 0.49 \\ 0.03 & -0.06 & 0.24 \\ 0.14 & -0.07 & -0.40 \\ 0.16 & -0.08 & -0.13 \\ 0.31 & -0.19 & -0.53 \\ 0.14 & -0.02 & 0.13 \\ 0.14 & -0.02 & -0.42 \\ 0.27 & -0.10 & -0.51 \end{bmatrix}, \text{bias}(\hat{\Psi}) \\ &= \begin{bmatrix} -0.42 \\ 0.38 & -0.46 \end{bmatrix}, \phi_1 = 1.00, \phi_2 \\ &= .97, \phi_3 = .56. \end{aligned} \quad (19)$$

The changes in outcome values from Equation 16 (where equivalent form $\eta^{7'} = [\eta_1^1 \quad -\eta_2^1 \quad -\eta_3^1]$ was followed) to Equation 19 (where equivalent form $\eta^{1'} = [\eta_1^1 \quad \eta_2^1 \quad \eta_3^1]$

was followed) illustrate the potential usefulness in re-signing parameter estimates. The average value within $\text{bias}(\hat{\Lambda})$ changed from -0.15 in Equation 16 to -0.02 in Equation 19. Average MSE with regard to $\hat{\Lambda}$ changed from 0.42 in Equation 16 to 0.06 in Equation 19. The average value within $\text{bias}(\hat{\Psi})$ changed from $-.64$ in Equation 16 to $-.17$ in Equation 19. Average MSE with regard to $\hat{\Psi}$ changed from 0.44 in Equation 16 to 0.18 in Equation 19. The K value changed from $-.18$ in Equation 16 to $.84$ in Equation 19. In summary, manual identification (e.g., Equation 15) and remediation (i.e., Equation 17 and Equation 18) of an equivalent form of η might be both useful and easy to accomplish when working with a single data set.

IDENTIFICATION AND REMEDIATION OF AN EQUIVALENT FORM OF η : MULTIPLE DATASETS

Identification and remediation of an equivalent form of η might be useful but difficult to accomplish manually when working with multiple data sets (e.g., replications), particularly when the replications do not uniformly follow a single equivalent form of η . For example, manual identification might have to occur separately for each of the many replications often observed in simulation studies with EFA. Remediation, too, might have to be determined separately for each replication (e.g., Equation 17 and Equation 18 would be appropriate only when equivalent form $\eta^{7'}$ is observed). In addition to being inefficient, manual identification or remediation of an equivalent form of η with multiple data sets might be error-prone for even a single data set as I increases rapidly as m increases (e.g., when $m = 3, 5, 7, I = 48, 3,840, 645,120$, respectively).

Despite the potential consequences related to the possible presence of equivalent forms of η in simulation studies with EFA, a general algorithm within freely available software has yet to emerge for addressing the possibility that a nonuniform order or sign pattern within η could be observed across replications. The lack of a general and accessible algorithm for addressing the possible presence of equivalent forms of η likely has contributed to the absence of any description for how this issue was handled in many published simulation studies with EFA. Brief descriptions of innovative study-specific algorithms can, however, be found in the literature. For example, Moore, Reise, Depaoli, and Haviland (2015) described their approach as follows: “To ensure that the RMSEs [root mean squared errors] in the present study were calculated using the same factor order for both the population and exploratory solutions, they were calculated using every possible re-ordering of the exploratory solution and the lowest RMSE was taken to be the one corresponding to the most congruent re-ordering” (p. 153). A brief description of the approach taken in de Winter et al. (2009) is as follows: “To recover order and sign of the loadings, the K s

[congruence coefficients] for the factor combinations ($f \times f$) between the sample solution and the population solution were calculated” (p. 152). A natural limitation of a study-specific algorithm is the inability to provide strong evidence for the efficacy of the algorithm within the manuscript for that study due both to space limitations and to an unknown order or sign pattern within $\boldsymbol{\eta}$ for each replication. Another likely limitation of a study-specific algorithm is the difficulty encountered when trying to use the algorithm in a subsequent study (e.g., previous code is unavailable, a different algorithm is needed, etc.).

Some software programs provide access to a postrotation algorithm for the identification and remediation of an equivalent form of $\boldsymbol{\eta}$. For example, the comprehensive EFA software (CEFA; Browne, Cudek, Tateneni, & Mels, 2010) allows a user to specify an order matrix unless target rotation is used. As Browne et al. (2010) explained, “an order matrix and a target matrix have the same appearance but different functions” (p. 6). An order matrix addresses the possible presence of an equivalent form of $\boldsymbol{\eta}$ whereby “columns of the previously rotated factor matrix are reordered and reflected (all elements of the column multiplied by -1) so as to minimize the sum of squares differences of rotated elements and corresponding specified elements of the order matrix” (Browne et al., p. 6). Elements in an order matrix are either unspecified or specified. A specified element informs the algorithm and is given a value (e.g., 0) or is required to be positive (and is denoted by a $+$ symbol). If a specified value within a column is unequal to 0 then a $+$ specification is disallowed in that column. A maximum of one $+$ element should be specified per column. Within any column, if each specified value is 0 and no element is specified as $+$, then that column is reflected if needed to achieve a positive sum of elements. An order matrix can therefore be viewed as innovative but limited information (e.g., unspecified elements, constraints on specified elements within each column) postrotation algorithm for the identification and remediation of an equivalent form of $\boldsymbol{\eta}$. Potential consequences of relying exclusively on an order/target matrix approach were mentioned by Browne et al.: “hopefully the corresponding element of the rotated matrix will be close to .3 [the value specified] but it could be negative [and cause an undesirable reflection of a column of the rotated factor matrix]” (p. 7).

An approach to addressing the possibility that a nonuniform order or sign pattern within $\boldsymbol{\eta}$ might be observed across replications that is similar to the approach described in this article was provided by Zhang, Preacher, and Luo (2010). One difference between the approach taken by Zhang et al. and the approach taken in this study is that the current approach can handle simulation studies with fuller parameterizations of ESEM (i.e., Example 2 in this article), whereas the previous approach appears to be focused on EFA only (i.e., Example 1 in this article). A second difference between the approach taken by Zhang

et al. and the approach taken in this article is that the current approach provides an R package for use in future research, whereas the previous approach provides an important, but perhaps somewhat less user-friendly, link to some R code (p. 111).

The approaches reviewed for the identification and remediation of an equivalent form of $\boldsymbol{\eta}$ informed the algorithm implemented in REREFAC. First, the presence of an a priori theory will be used in Step 3 to help identify which equivalent form of $\boldsymbol{\eta}$ each replication follows. Second, the option of changing the sign pattern or the order within $\boldsymbol{\eta}$ (and within arrays of model parameters affected by an undesirable sign pattern within $\boldsymbol{\eta}$) will be used in Step 4 as a remedy when a replication is identified as following an undesired equivalent form of $\boldsymbol{\eta}$.

REREFAC ALGORITHM

Step 3 of the algorithm is to determine which equivalent form of $\boldsymbol{\eta}$ (within the set of $\boldsymbol{\eta}^i$ indexed in Step 2) each replication follows. For each replication, average *MSE* with regard to $\hat{\Lambda}$ is computed I times to account for each possible equivalent form of the relevant population values based on both the complete a priori theory, Λ , and the relevant set of \mathbf{P}^i indexed in Step 2 (see Equation 6). Simply, and for each replication, $\hat{\Lambda}$ does not change in Step 3 but the Λ that it is compared to does change I times. The smallest average *MSE* value determines which $\boldsymbol{\eta}^i$ a replication follows.⁷

Step 4 of the algorithm is to use the appropriate \mathbf{P}^i to reorder or re-sign parameter estimates so that all replications are made to uniformly follow the order and sign pattern defined by the population values. More specifically, within each replication parameter estimates (now indexed with an i superscript to communicate the determination made in Step 3) are re-signed or reordered to follow the desired equivalent form, $\boldsymbol{\eta}^{1'} = [\eta_1^1 \ \eta_2^1 \ \eta_3^1]$:

$$\hat{\Lambda}^i \mathbf{P}^i = \hat{\Lambda}^1 \quad (20)$$

$$\mathbf{P}^{i'} \hat{\Psi}^i \mathbf{P}^i = \hat{\Psi}^1. \quad (21)$$

Note that if a replication is determined to follow the desired form, $\boldsymbol{\eta}^{1'} = [\eta_1^1 \ \eta_2^1 \ \eta_3^1]$, in Step 3 then Equation 20 and Equation 21 can be rewritten:

$$\hat{\Lambda}^1 \mathbf{I} = \hat{\Lambda}^1 \quad (22)$$

$$\mathbf{I} \hat{\Psi}^1 \mathbf{I} = \hat{\Psi}^1, \quad (23)$$

⁷In the event of a tie, the largest K value determines which $\boldsymbol{\eta}^i$ a replication follows.

which maintains the order and sign pattern observed in the rotated solution.

REREFAC T RESULTS

Based on Example 1 a simulation study was designed to test the efficacy of the algorithm implemented in REREFAC T to identify and remediate equivalent forms of $\boldsymbol{\eta}$. One hundred data sets, where $N = 300$, were drawn from the multivariate normal distribution $(\mathbf{0}, \boldsymbol{\Sigma})$ described previously. These data sets were used to produce 4,800 replications of $\hat{\Lambda}$ and $\hat{\Psi}$ within an ESEM framework under ML estimation in *Mplus* 7.31. The default rotation criterion, geomin (Yates, 1987), was used with ε set to equal .50 (e.g., Marsh et al., 2010). More specifically, the 100 data sets were fit to a three-factor EFA 48 times where the parameter values in the model statement were altered each time to determine the equivalent form of $\boldsymbol{\eta}$ that each of the 4,800 replications of $\hat{\Lambda}$ and $\hat{\Psi}$ followed (e.g., replications 1–100 followed equivalent form $\boldsymbol{\eta}^1$, replications 101–200 followed equivalent form $\boldsymbol{\eta}^2$, etc.).

The equivalent form of $\boldsymbol{\eta}$ followed by each of the 4,800 replications was correctly identified in Step 3 (e.g., replications 1–100 were identified as following equivalent form $\boldsymbol{\eta}^1$, replications 101–200 were identified as following equivalent form $\boldsymbol{\eta}^2$, etc.). Table 4 provides some outcome values for each of the 48 equivalent forms of $\boldsymbol{\eta}$ with sample data pre- and postimplementation of Step 4. The pattern of results observed in Table 4 for the preimplementation of Step 4 is very similar to the pattern observed previously with the population data (see Table 3).⁸ The pattern of results observed in Table 4 postimplementation of Step 4 is consistent with extrapolating the results from the first row of outcome values, $\boldsymbol{\eta}^1$, before the implementation of Step 4, to the remaining rows of outcome values, $\boldsymbol{\eta}^2$ to $\boldsymbol{\eta}^{48}$, postimplementation of Step 4.

SIMULATION STUDIES WITH ESEM

Indeterminacies within $\boldsymbol{\eta}$ have implications for additional arrays of model parameters in fuller parameterizations of ESEM. The reorganization or re-signing of model parameters that could be expected given the indeterminacies within $\boldsymbol{\eta}$ can be demonstrated through a formal presentation of the model. The measurement model in ESEM can be written as:

$$\mathbf{y}_{pxl} = \mathbf{v}_{pxl} + \Lambda_{pxm}\boldsymbol{\eta}_{mxl} + \mathbf{K}_{pxq}\mathbf{x}_{qx1} + \boldsymbol{\varepsilon}_{pxl}, \text{ where} \quad (24)$$

q is the number of observed exogenous variables,
 \mathbf{v} is a vector of intercepts or means,
 \mathbf{K} is a matrix of (\mathbf{y} on \mathbf{x}) regression coefficients, and
 \mathbf{x} is a vector of observed independent variables.

The latent variable model in ESEM can be written as:

$$\boldsymbol{\eta}_{mxl} = \boldsymbol{\alpha}_{mxl} + \mathbf{B}_{mxm}\boldsymbol{\eta}_{mxl} + \boldsymbol{\Gamma}_{mxq}\mathbf{x}_{qx1} + \boldsymbol{\zeta}_{mxl}, \text{ where} \quad (25)$$

$\boldsymbol{\alpha}$ is a vector of intercepts or means,
 \mathbf{B} is a matrix of ($\boldsymbol{\eta}$ on $\boldsymbol{\eta}$) regression coefficients, and
 $\boldsymbol{\Gamma}$ is a matrix of ($\boldsymbol{\eta}$ on \mathbf{x}) regression coefficients.

Replacing $\boldsymbol{\eta}$ with $\mathbf{P}^i\boldsymbol{\eta}$ in Equation 24 and Equation 25 leads to the following expressions that describe the reorganization or re-signing of additional model parameters that could be expected given the indeterminacies within $\boldsymbol{\eta}$:

$$\boldsymbol{\alpha}^i = \mathbf{P}^i\boldsymbol{\alpha} \quad (26)$$

$$\mathbf{B}^i = \mathbf{P}^i\mathbf{B}\mathbf{P}^{i'} \quad (27)$$

$$\boldsymbol{\Gamma}^i = \mathbf{P}^i\boldsymbol{\Gamma}. \quad (28)$$

Equation 6 with regard to Λ^i and Equation 7 with regard to Ψ^i are unchanged.

REREFAC T ALGORITHM

When all endogenous variables are in the EFA, then Step 1 through Step 4 proceed as previously described with the exception of reading in additional arrays of parameter estimates to be reordered or re-signed in Step 4. More specifically, within each replication parameter estimates are re-signed or reordered to follow the desired equivalent form, $\boldsymbol{\eta}^1$, as follows:

$$\mathbf{P}^{i'}\hat{\boldsymbol{\alpha}}^i = \hat{\boldsymbol{\alpha}}^1 \quad (29)$$

$$\mathbf{P}^{i'}\hat{\mathbf{B}}^i\mathbf{P}^i = \hat{\mathbf{B}}^1 \quad (30)$$

$$\mathbf{P}^{i'}\hat{\boldsymbol{\Gamma}}^i = \hat{\boldsymbol{\Gamma}}^1. \quad (31)$$

Equation 20 with regard to $\hat{\Lambda}^1$ and Equation 21 with regard to $\hat{\Psi}^1$ are unchanged.

When all of the endogenous variables in the ESEM are not in the EFA then Step 1 and Step 4 proceed somewhat differently because each array that could be expected to be affected by indeterminacies within $\boldsymbol{\eta}$ is a partitioned array. The partitioning of an array serves to identify elements that would be (un)affected by indeterminacies within $\boldsymbol{\eta}$. A vector

⁸The absolute value for outcomes in regard to $\hat{\Psi}$ tend to be larger in Table 4 than in Table 3 because in Table 4 the diagonal elements are not model parameters.

TABLE 4
Some Outcome Values for Equivalent Forms of the Vector of Factors, η [η_1 ; η_2 ; η_3], From Example 1 With Sample Data: Pre- and Post-Step 4 in REREFAC T

η^i	Pre-REREFAC T					Post-REREFAC T				
	Average Bias		Average MSE		K	Average Bias		Average MSE		K
	$\hat{\Lambda}$	$\hat{\Psi}$	$\hat{\Lambda}$	$\hat{\Psi}$	$\hat{\Lambda}$	$\hat{\Lambda}$	$\hat{\Psi}$	$\hat{\Lambda}$	$\hat{\Psi}$	$\hat{\Lambda}$
η^1	0.02	-0.08	0.01	0.03	0.98	0.02	-0.08	0.01	0.03	0.98
η^2	-0.20	-0.84	0.30	0.62	0.32	0.02	-0.08	0.01	0.03	0.98
η^3	-0.18	-0.59	0.26	0.46	0.33	0.02	-0.08	0.01	0.03	0.98
η^4	-0.40	-0.49	0.55	0.19	-0.33	0.02	-0.08	0.01	0.03	0.98
η^5	-0.11	-0.49	0.14	0.19	0.33	0.02	-0.08	0.01	0.03	0.98
η^6	-0.33	-0.59	0.44	0.46	-0.33	0.02	-0.08	0.01	0.03	0.98
η^7	-0.31	-0.84	0.39	0.62	-0.32	0.02	-0.08	0.01	0.03	0.98
η^8	-0.53	-0.08	0.68	0.03	-0.98	0.02	-0.08	0.01	0.03	0.98
η^9	0.02	-0.08	0.20	0.06	0.31	0.02	-0.08	0.01	0.03	0.98
η^{10}	-0.20	-0.84	0.50	0.59	-0.35	0.02	-0.08	0.01	0.03	0.98
η^{11}	-0.18	-0.59	0.19	0.27	0.37	0.02	-0.08	0.01	0.03	0.98
η^{12}	-0.40	-0.49	0.48	0.38	-0.29	0.02	-0.08	0.01	0.03	0.98
η^{13}	-0.11	-0.49	0.21	0.38	0.29	0.02	-0.08	0.01	0.03	0.98
η^{14}	-0.33	-0.59	0.51	0.27	-0.37	0.02	-0.08	0.01	0.03	0.98
η^{15}	-0.31	-0.84	0.19	0.59	0.35	0.02	-0.08	0.01	0.03	0.98
η^{16}	-0.53	-0.08	0.49	0.06	-0.31	0.02	-0.08	0.01	0.03	0.98
η^{17}	0.02	-0.08	0.31	0.14	0.10	0.02	-0.08	0.01	0.03	0.98
η^{18}	-0.20	-0.84	0.34	0.47	0.02	0.02	-0.08	0.01	0.03	0.98
η^{19}	-0.18	-0.59	0.29	0.18	0.17	0.02	-0.08	0.01	0.03	0.98
η^{20}	-0.40	-0.49	0.32	0.51	0.09	0.02	-0.08	0.01	0.03	0.98
η^{21}	-0.11	-0.49	0.37	0.51	-0.09	0.02	-0.08	0.01	0.03	0.98
η^{22}	-0.33	-0.59	0.40	0.18	-0.17	0.02	-0.08	0.01	0.03	0.98
η^{23}	-0.31	-0.84	0.35	0.47	-0.02	0.02	-0.08	0.01	0.03	0.98
η^{24}	-0.53	-0.08	0.38	0.14	-0.10	0.02	-0.08	0.01	0.03	0.98
η^{25}	0.02	-0.08	0.18	0.20	0.46	0.02	-0.08	0.01	0.03	0.98
η^{26}	-0.20	-0.84	0.20	0.37	0.38	0.02	-0.08	0.01	0.03	0.98
η^{27}	-0.18	-0.59	0.43	0.28	-0.19	0.02	-0.08	0.01	0.03	0.98
η^{28}	-0.40	-0.49	0.45	0.45	-0.27	0.02	-0.08	0.01	0.03	0.98
η^{29}	-0.11	-0.49	0.24	0.45	0.27	0.02	-0.08	0.01	0.03	0.98
η^{30}	-0.33	-0.59	0.26	0.28	0.19	0.02	-0.08	0.01	0.03	0.98
η^{31}	-0.31	-0.84	0.49	0.37	-0.38	0.02	-0.08	0.01	0.03	0.98
η^{32}	-0.53	-0.08	0.51	0.20	-0.46	0.02	-0.08	0.01	0.03	0.98
η^{33}	0.02	-0.08	0.29	0.18	0.15	0.02	-0.08	0.01	0.03	0.98
η^{34}	-0.20	-0.84	0.31	0.39	0.07	0.02	-0.08	0.01	0.03	0.98
η^{35}	-0.18	-0.59	0.37	0.47	-0.04	0.02	-0.08	0.01	0.03	0.98
η^{36}	-0.40	-0.49	0.40	0.26	-0.12	0.02	-0.08	0.01	0.03	0.98
η^{37}	-0.11	-0.49	0.30	0.26	0.12	0.02	-0.08	0.01	0.03	0.98
η^{38}	-0.33	-0.59	0.32	0.47	0.04	0.02	-0.08	0.01	0.03	0.98
η^{39}	-0.31	-0.84	0.38	0.39	-0.07	0.02	-0.08	0.01	0.03	0.98
η^{40}	-0.53	-0.08	0.40	0.18	-0.15	0.02	-0.08	0.01	0.03	0.98
η^{41}	0.02	-0.08	0.22	0.09	0.46	0.02	-0.08	0.01	0.03	0.98
η^{42}	-0.20	-0.84	0.25	0.52	0.38	0.02	-0.08	0.01	0.03	0.98
η^{43}	-0.18	-0.59	0.31	0.56	0.26	0.02	-0.08	0.01	0.03	0.98
η^{44}	-0.40	-0.49	0.34	0.13	0.19	0.02	-0.08	0.01	0.03	0.98
η^{45}	-0.11	-0.49	0.35	0.13	-0.19	0.02	-0.08	0.01	0.03	0.98
η^{46}	-0.33	-0.59	0.38	0.56	-0.26	0.02	-0.08	0.01	0.03	0.98
η^{47}	-0.31	-0.84	0.44	0.52	-0.38	0.02	-0.08	0.01	0.03	0.98
η^{48}	-0.53	-0.08	0.47	0.09	-0.46	0.02	-0.08	0.01	0.03	0.98

Note. The indexing of η^i follows the indexing established in Table 2. MSE = mean square error; K = congruence.

of endogenous variables in the ESEM but not in the EFA is denoted with a 0 superscript indicating that it would be unaffected by indeterminacies within η (e.g., η^0). The general form of a partitioned η is:

$$\eta = \begin{bmatrix} \eta_{m^1 \times 1}^1 \\ \eta_{m^0 \times 1}^0 \end{bmatrix}, \text{ where} \quad (32)$$

m^1 is the number of η_k^1 in the EFA, m^0 is the number of η_k^0 not in the EFA, and $m^1 + m^0 = m$ (i.e., the total number of η_k in the ESEM).

Step 1 through Step 3 proceed as previously described with the exception that η^0 within η is not counted when determining I in Step 1. Step 4 proceeds somewhat differently due to the partitioned η . The general form of each array affected by indeterminacies within η is:

$$\alpha = \begin{bmatrix} \alpha_{m^1 \times 1}^{\eta^1} \\ \alpha_{m^0 \times 1}^{\eta^0} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{0}_{m^1 \times m^1}^{\eta^1, \eta^1} \mathbf{B}_{m^1 \times m^0}^{\eta^1, \eta^0} \\ \mathbf{B}_{m^0 \times m^1}^{\eta^0, \eta^1} \mathbf{B}_{m^0 \times m^0}^{\eta^0, \eta^0} \end{bmatrix}, \Gamma = \begin{bmatrix} \Gamma_{m^1 \times q}^{\eta^1} \\ \Gamma_{m^0 \times q}^{\eta^0} \end{bmatrix}, \Lambda = \begin{bmatrix} \Lambda_{p^1 \times m^1}^{\eta^1, \eta^1} \Lambda_{p^1 \times m^0}^{\eta^1, \eta^0} \\ \mathbf{0}_{p^0 \times m^1}^{\eta^0, \eta^1} \Lambda_{p^0 \times m^0}^{\eta^0, \eta^0} \end{bmatrix}, \Psi = \begin{bmatrix} \Psi_{m^1 \times m^1}^{\zeta^1, \zeta^1} \Psi_{m^1 \times m^0}^{\zeta^1, \zeta^0} \\ \Psi_{m^0 \times m^1}^{\zeta^0, \zeta^1} \Psi_{m^0 \times m^0}^{\zeta^0, \zeta^0} \end{bmatrix}. \quad (33)$$

Superscripts identify parts of an array that would be (un) affected by indeterminacies within η .

For example, α^{η^1} and Γ^{η^1} are affected, whereas α^{η^0} and Γ^{η^0} are unaffected. Superscripts within \mathbf{B} identify partitions that are affected, $\mathbf{B}^{\eta^1, \eta^0}$ (i.e., η^1 regressed on η^0) and $\mathbf{B}^{\eta^0, \eta^1}$, and are unaffected, $\mathbf{B}^{\eta^0, \eta^0}$, whereas $\mathbf{0}^{\eta^1, \eta^1}$ imposes a model-based restriction (i.e., η^1 cannot be regressed on η^1). Superscripts within Λ identify partitions that are affected, Λ^{η^1, η^1} (i.e., \mathbf{y}^1 regressed on η^1), and are unaffected, Λ^{η^0, η^0} and $\mathbf{0}^{\eta^0, \eta^0}$, whereas $\mathbf{0}^{\eta^0, \eta^1}$ imposes a model-based restriction (e.g., \mathbf{y}^0 cannot be regressed on η^1 because \mathbf{y}^0 is not in the EFA).

To use Equations 20, 21, and 30 to 32 in Step 4 to reorder or re-sign parameter estimates affected by indeterminacies within η , \mathbf{P}^i is placed within a partitioned matrix. Placing \mathbf{P}^i within a partitioned matrix allows the aforementioned function to be accomplished while not reordering or re-signing elements unaffected by indeterminacies within η . The general form of the partitioned matrix within which \mathbf{P}^i is placed is:

$$\begin{bmatrix} \mathbf{P}_{m^1 \times m^1}^i \mathbf{0}_{m^1 \times m^0} \\ \mathbf{0}_{m^0 \times m^1} \mathbf{I}_{m^0 \times m^0} \end{bmatrix}. \quad (34)$$

For convenience this entire partitioned array is referred to as \mathbf{P}^i .

REREFAC T RESULTS

Example 2 extended Example 1 by adding η_4^0 to the ESEM while excluding it from the EFA. Thus, in Step 1 the desired form from which order or sign changes could occur in an equivalent form of $\boldsymbol{\eta}'$ was: $[\eta_1^1 \eta_2^1 \eta_3^1 \eta_4^0]$. Because only three of the four η_k were in the EFA, $I = 48$, as in Example 1. Step 2 proceeded as previously described to index within I . Thus, Table 2 still provides each of the 48 \mathbf{P}^i and corresponding $\boldsymbol{\eta}^i$.

Example 2 extended Example 1 in three additional ways. First, an exogenous observed variable, x_1 ($M = 0, SD = 1$), was added to the model and exerted a direct effect on $\eta_1^1(.50)$, $\eta_2^1(.25)$, and $\eta_3^1(.00)$. Percentage of variance accounted for ranged from 0% to 20% for η_1^1 through η_3^1 . Second, an intercept was specified for $\eta_1^1(-.25)$, $\eta_2^1(.25)$, and $\eta_3^1(.00)$. Third, η_4^0 was specified to receive a direct effect from $\eta_1^1(.50)$, $\eta_2^1(.25)$, and $\eta_3^1(.00)$. Parameter arrays unique to Example 2 and with values following the desired equivalent form, $\boldsymbol{\eta}^1$, were:

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \begin{bmatrix} -.25 \\ .25 \\ .00 \\ .00 \end{bmatrix}, \mathbf{B}^1 = \begin{bmatrix} .00.00.00.00 \\ .00.00.00.00 \\ .00.00.00.00 \\ .50.25.00.00 \end{bmatrix}, \boldsymbol{\Gamma}^1 \\ &= \begin{bmatrix} .50 \\ .25 \\ .00 \\ .00 \end{bmatrix}. \end{aligned} \tag{35}$$

Parameter arrays carried over from Example 1 that were altered in Example 2 were:

$$\begin{aligned} \boldsymbol{\Lambda}^1 &= \begin{bmatrix} -.070.030.770.00 \\ 0.35-.030.420.00 \\ 0.050.87-.200.00 \\ -.160.650.210.00 \\ 0.770.030.100.00 \\ 0.360.250.060.00 \\ 0.72-.050.260.00 \\ 0.330.77-.060.00 \\ 0.630.01-.070.00 \\ 0.77-.05-.080.00 \\ 0.000.000.001.00 \end{bmatrix}, \boldsymbol{\Psi}^1 \\ &= \begin{bmatrix} 1.00.50.25.00 \\ .501.00.00.00 \\ .25.001.00.00 \\ .00.00.00.62 \end{bmatrix}. \end{aligned} \tag{36}$$

Example 2 extended the simulation study for Example 1 to accommodate the extensions made in Example 2. The purpose of this second simulation, however, was identical to the purpose of the first simulation: to test the efficacy of the algorithm implemented in REREFAC T to identify

and remediate equivalent forms of $\boldsymbol{\eta}$. Parameter values were consistent with Equations 35 and 36. One hundred data sets, where $N = 300$, were drawn from a multivariate normal distribution. These data sets were used to produce 4,800 replications of $\hat{\boldsymbol{\alpha}}, \hat{\mathbf{B}}, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\Lambda}}$, and $\hat{\boldsymbol{\Psi}}$ within an ESEM framework under ML estimation. Geomin rotation was used with ε set to equal .50. More specifically, the 100 data sets were fit to a three-factor ESEM 48 times where the parameter values in the model statement were altered each time to determine the equivalent form of $\boldsymbol{\eta}$ that each of the 4,800 replications of $\hat{\boldsymbol{\Lambda}}$ and $\hat{\boldsymbol{\Psi}}$ followed.

The equivalent form of $\boldsymbol{\eta}$ followed by each of the 4,800 replications was correctly identified in Step 3 (e.g., replications 1–100 were identified as following equivalent form $\boldsymbol{\eta}^1$, replications 101–200 followed equivalent form $\boldsymbol{\eta}^2$, etc.). Table 5 provides average *MSE* for parameter estimates within each parameter array that was unique to Example 2 for each of the 48 equivalent forms of $\boldsymbol{\eta}$ with sample data before and after implementation of Step 4. The pattern of results observed in Table 5 preimplementation of Step 4 was consistent with expectations based on the parameter values in Equation 35 and the reorganization or re-signing of these parameter values that could be expected (see Equations 26–28) given the indeterminacies within $\boldsymbol{\eta}$. The pattern of results observed in Table 5 postimplementation of Step 4 was consistent with extrapolating the results from the first row of outcome values, $\boldsymbol{\eta}^1$, preimplementation of Step 4, to the remaining rows of outcome values, $\boldsymbol{\eta}^2$ – $\boldsymbol{\eta}^{48}$, postimplementation of Step 4.

HOW TO USE REREFAC T: A BRIEF DEMONSTRATION

Using REREFAC T is briefly demonstrated in this section by providing annotations for key commands and condensed output using a subset of simulated data from Example 1. The data analyzed in this section were provided by replications 1 through 100 (i.e., following equivalent form $\boldsymbol{\eta}^1$) and 4,701 through 4,800 (i.e., following equivalent form $\boldsymbol{\eta}^{48}$). Complete commands for this demonstration are provided in Appendix A. Complete commands for a subset of simulated data (also from replications 1–100 and 4,701–4,800) from Example 2 are provided in Appendix B. The reference manual provides a definition for each data object, function, argument, and value and is available at <https://cran.r-project.org/web/packages/REREFAC T/>.

The function provided here installs the REREFAC T package within R (R Development Core Team, 2015).

```
> install.packages("REREFAC T", dependencies = TRUE)
```


TABLE 5
Some Outcome Values for Equivalent Forms of the Vector of Factors,
 η [η_1 ; η_2 ; η_3], From Example 2 With Sample Data: Pre- and Post-Step
4 in REREFACT

η^i	Pre-REREFACT			Post-REREFACT		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\Gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\Gamma}$
η^1	0.01	0.00	0.00	0.01	0.00	0.00
η^2	0.03	0.08	0.09	0.01	0.00	0.00
η^3	0.03	0.03	0.02	0.01	0.00	0.00
η^4	0.05	0.11	0.11	0.01	0.00	0.00
η^5	0.01	0.00	0.00	0.01	0.00	0.00
η^6	0.03	0.08	0.09	0.01	0.00	0.00
η^7	0.03	0.03	0.02	0.01	0.00	0.00
η^8	0.05	0.11	0.11	0.01	0.00	0.00
η^9	0.02	0.01	0.01	0.01	0.00	0.00
η^{10}	0.04	0.09	0.09	0.01	0.00	0.00
η^{11}	0.02	0.01	0.01	0.01	0.00	0.00
η^{12}	0.04	0.09	0.09	0.01	0.00	0.00
η^{13}	0.02	0.02	0.01	0.01	0.00	0.00
η^{14}	0.04	0.09	0.10	0.01	0.00	0.00
η^{15}	0.02	0.02	0.01	0.01	0.00	0.00
η^{16}	0.04	0.09	0.10	0.01	0.00	0.00
η^{17}	0.04	0.03	0.03	0.01	0.00	0.00
η^{18}	0.02	0.07	0.07	0.01	0.00	0.00
η^{19}	0.04	0.03	0.03	0.01	0.00	0.00
η^{20}	0.02	0.07	0.07	0.01	0.00	0.00
η^{21}	0.04	0.04	0.04	0.01	0.00	0.00
η^{22}	0.02	0.08	0.08	0.01	0.00	0.00
η^{23}	0.04	0.04	0.04	0.01	0.00	0.00
η^{24}	0.02	0.08	0.08	0.01	0.00	0.00
η^{25}	0.02	0.03	0.04	0.01	0.00	0.00
η^{26}	0.02	0.03	0.04	0.01	0.00	0.00
η^{27}	0.04	0.06	0.06	0.01	0.00	0.00
η^{28}	0.04	0.06	0.06	0.01	0.00	0.00
η^{29}	0.02	0.05	0.05	0.01	0.00	0.00
η^{30}	0.02	0.05	0.05	0.01	0.00	0.00
η^{31}	0.04	0.07	0.07	0.01	0.00	0.00
η^{32}	0.04	0.07	0.07	0.01	0.00	0.00
η^{33}	0.04	0.02	0.03	0.01	0.00	0.00
η^{34}	0.04	0.02	0.03	0.01	0.00	0.00
η^{35}	0.02	0.08	0.07	0.01	0.00	0.00
η^{36}	0.02	0.08	0.07	0.01	0.00	0.00
η^{37}	0.04	0.03	0.04	0.01	0.00	0.00
η^{38}	0.04	0.03	0.04	0.01	0.00	0.00
η^{39}	0.02	0.08	0.08	0.01	0.00	0.00
η^{40}	0.02	0.08	0.08	0.01	0.00	0.00
η^{41}	0.05	0.01	0.01	0.01	0.00	0.00
η^{42}	0.03	0.05	0.05	0.01	0.00	0.00
η^{43}	0.03	0.06	0.05	0.01	0.00	0.00
η^{44}	0.01	0.10	0.10	0.01	0.00	0.00
η^{45}	0.05	0.01	0.01	0.01	0.00	0.00
η^{46}	0.03	0.05	0.05	0.01	0.00	0.00
η^{47}	0.03	0.06	0.05	0.01	0.00	0.00
η^{48}	0.01	0.10	0.10	0.01	0.00	0.00

Note. The indexing of η^i follows the indexing established in Table 2. MSE = mean square error.

Use of the REREFACT package depends on the R packages “psych”, “combinat”, “Matrix”, and “gdata”, which are automatically installed when REREFACT is installed. After all necessary packages have been installed, the R library function with the package argument can be specified as provided here to use REREFACT.

```
> library(REREFACT)
```

Data Management

Relevant data objects should be loaded. In this demonstration there were three relevant data objects that are all available in the package and can be called from the package. The Λ^1 (see Equation 3 for values) for this demonstration was loaded as a 10×3 matrix as depicted next (with the second line of code allowing the user to view the data object loaded).

```
> data(pop_L_efa)
> pop_L_efa
```

Replications of $\hat{\Lambda}$ should be loaded as a list object containing all $\hat{\Lambda}$ matrices. In this demonstration the data object `sample_lambda_efa` provided a list of the 200 replications of $\hat{\Lambda}$ and was loaded with the code provided here.

```
> data(sample_lambda_efa)
> sample_lambda_efa
```

Replications of $\hat{\psi}$ should be loaded as a list object containing all $\hat{\psi}$ matrices. In this demonstration the data object `sample_psi_efa` provided a list of the 200 replications of $\hat{\psi}$ and was loaded with the code provided here.

```
> data(sample_psi_efa)
> sample_psi_efa
```

After the data objects have been loaded, REREFACT uses two functions, `rerefact()` and `correct_XXXX()`, to implement the four-step algorithm previously described in this article.

Steps 1 Through 3

The `rerefact()` function implements Steps 1 through 3 of the algorithm implemented within the REREFACT R package. Arguments within this function to accomplish Steps 1 through 3 for this demonstration were specified as provided here.


```
> rerefact(n.factor = 3, n.eta = 3, n.var = 10, pop_lambda =
pop_L_efa, sample_lambda = sample_lambda_efa)
```

The n.factor argument refers to the number of latent variables within η affected by indeterminacies within η (i.e., m^1) and in this demonstration equaled three. The n.eta argument refers to the total number of latent variables within η (i.e., m) and in this demonstration equaled three (i.e., $m = m^1$). The n.var argument refers to the number of observed endogenous variables with regard to n.factor and in this demonstration equaled 10. The pop_lambda argument refers to the population pattern coefficient matrix (i.e., Λ^1) loaded as a matrix and in this demonstration equaled the data object pop_L_efa previously described. The sample_lambda argument refers to the replications of $\hat{\Lambda}$ loaded as a list of matrices and equaled the data object sample_lambda_efa previously described.

Three key values are provided by the rerefact() function: n.perm, permutation, and correct.permutation. The n.perm value refers to the total number of equivalent forms of η that can result from indeterminacies within η (i.e., I) and provides the result for Step 1 of the algorithm. For this demonstration the n.perm value equaled 48 and was output as depicted here.

```
$n.perm
[1] 48
```

Data in the first column simply denotes row of the output (e.g., [1] = row 1).

The permutation value refers to a matrix that indexes within I via a unique, orthogonal permutation matrix for each equivalent form of η and provides the result for Step 2 of the algorithm. For this demonstration the permutation value yielded $\eta^{1,2,\dots,48}$ and $P^{1,2,\dots,48}$ and was output as depicted (condensed for spatial reasons) here.

```
$permutation
permutation index order1 order2 order3 sign1 sign2 sign3
[1,] 1 1 2 3 1 1 1
[2,] 2 1 2 3-1 1 1
....
[48,] 48 2 1 3-1-1-1
```

The permutation index column refers to the indexing within I . The order1, order2, order3 columns refer to the ordering of $\eta_1^1, \eta_2^1, \eta_3^1$, respectively. The sign1, sign2, sign3, columns refer to the sign of $\eta_1^1, \eta_2^1, \eta_3^1$, respectively. For

example, permutation index = 1 refers to $\eta^1 = \begin{bmatrix} \eta_1^1 \\ \eta_2^1 \\ \eta_3^1 \end{bmatrix}$ and

$P^1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, whereas permutation index = 48 refers to

$$\eta^{48} = \begin{bmatrix} -\eta_2^1 \\ -\eta_1^1 \\ -\eta_3^1 \end{bmatrix} \text{ and } P^{48} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The \$permutation output provided the 48 P^i and η^i depicted in the first two columns of Table 2.

The correct.permutation value refers to a matrix that provides the specific equivalent form of η (within the set of η^i indexed in Step 2) that each replication follows and provides the result for Step 3 of the algorithm. For this demonstration, the correct.permutation value yielded a table with a number of rows equal to the number of replications (200 in this case) and was output as depicted (condensed for spatial reasons) here.

```
$correct.permutation
rep# order1 order2 order3 sign1 sign2 sign3 correct_perm
Bias MSE
1 1 2 3 1 1 1 1 0.02 0.01
2 1 2 3 1 1 1 1 0.03 0.01
....
199 2 1 3-1-1 -1 48-0.05 0.01
200 2 1 3-1 -1 -1 48-0.01 0.01
```

The rep# column provides a number for each replication of $\hat{\Lambda}$ loaded in the sample_lambda argument. The next six columns are defined as previously described. The correct_perm column refers to the specific equivalent form of η (within the set of η^i indexed in Step 2) that a given replication follows. The Bias column refers to the average value within $bias(\hat{\Lambda})$ for a given replication when compared to the specific equivalent form of Λ^i identified in correct_perm (e.g., Λ^1 for repetition 1). The MSE column refers to the average MSE with regard to $\hat{\Lambda}$ for a given replication when compared to the specific equivalent form of Λ^i identified in correct_perm. (e.g., Λ^{48} for repetition 200).

By the conclusion of Step 3 a text file, P.csv, is automatically created and saved to the working directory. In case users don't know their working directory or did not previously set it themselves, the users can run

```
> getwd()
```

to learn the path to the working directory where the files are saved. The P.csv file contains a list of the correct P^i for each replication. The P.csv file is used in Step 4 to reorder or re-sign the relevant parameter estimates within each replication so that all replications uniformly follow the order and sign pattern defined by the population values. For this demonstration the P.csv file was renamed P_efa. To use the P.csv file in Step 4, the following code imported the P.csv file from the working directory and assigned it to an R object named "P_efa".

```
> P_efa <- read.csv("P.csv")
> P_efa
```

Step 4

The `correct_XXXX()` function implements Step 4 of the algorithm implemented within the REREFACt R package. The XXXX within the `correct_XXXX()` function is a placeholder for text to be provided by the user to identify a parameter array that might be affected by indeterminacies within $\boldsymbol{\eta}$. In this demonstration parameter arrays that could be affected by indeterminacies within $\boldsymbol{\eta}$ included `lambda` and `psi`.

The `correct_lambda()` function reorders or re-signs as needed within $\hat{\Lambda}$ for each replication. Arguments within this function to accomplish Step 4 in this demonstration were specified as provided here.

```
> correct_lambda(P_data = P_efa, rep = 200, n.eta = 3,
sample_lambda = sample_lambda_efa)
```

The `P_data` argument refers to the correct \mathbf{P}^i for each replication and in this demonstration equaled the `P_efa` file obtained in Step 3. The `rep` argument refers to the number of replications and in this demonstration equaled 200. The final three arguments are defined as previously described. The `correct_lambda()` function returns a matrix, `correct_lambda`, of the reordered or re-signed estimated pattern coefficient matrix for each replication (indexed in the first column of `correct_lambda`) and saves the matrix as a text file to the designated working directory. The `correct_lambda` text file was used to provide the values depicted in the Average Bias $\hat{\Lambda}$ column and the Average *MSE* $\hat{\Lambda}$ column under Post-REREFACt in Table 4.

The `correct_psi()` function reorders or re-signs as needed within $\hat{\Psi}$ for each replication. Arguments within this function to accomplish Step 4 in this demonstration were specified as provided here.

```
> correct_psi(P_data = P_efa, rep = 200, n.eta = 3,
sample_psi = sample_psi_efa)
```

The first four arguments are defined as previously described. The `sample_psi` argument refers to the replications of $\hat{\Psi}$ loaded as a list of matrices and this demonstration equaled the data object `sample_psi_efa` previously described. The `correct_psi()` function returns a matrix, `correct_psi`, of the reordered or re-signed estimated covariance matrix for the vector of residuals for `eta` for each replication (indexed in the first column of `correct_psi`) and saves the matrix as a text file to the designated working directory. The `correct_psi` text file was used to provide the information depicted in the Average Bias $\hat{\Psi}$ column and the Average *MSE* $\hat{\Psi}$ column under Post-REREFACt in Table 4.

DISCUSSION

The purpose of this article was to describe the REREFACt R package, which provides a postrotation algorithm that reorders or reflects factors for each replication of a simulation study with EFA. REREFACt is important because

simulation studies that combine parameter estimates across replications where the factors do not uniformly follow the order or sign pattern defined by the population values could yield misleading results for outcomes commonly of interest in these studies (e.g., Table 3). Two simulation studies provided evidence for the efficacy of the REREFACt R package to identify and remediate equivalent forms of $\boldsymbol{\eta}$ in models with EFA only (i.e., Example 1) and in fuller parametrizations of ESEM (i.e., Example 2). Findings are discussed with respect to some existing postrotation algorithms for the identification and remediation of an equivalent form of $\boldsymbol{\eta}$.

Mplus uses parameter values, λ_{0ij} , provided by the user in the model statement to define an alignment criteria that minimizes the target function, $\sum (\lambda_{0ij} - s_j \lambda_{i\sigma(j)})^2$ over all factor permutations (σ) and sign assignments ($s_j = 1$ or -1) in ESEM (see Asparouhov & Muthén, 2009, p. 437). This innovative and general approach was used successfully in this study to control the order and sign pattern within $\boldsymbol{\eta}$ for data generation in Example 1 and Example 2 under geomin rotation. When the same approach was used under a target rotation, however, we were unsuccessful in controlling the order and sign pattern within $\boldsymbol{\eta}$ for data generation (see Equation 15). We suspect that the limited information supplied in the target matrix might have played an important role in the failure and that this approach could work well in most cases.

A variety of software programs have been used (e.g., CEFA in MacCallum et al., 2001; *Mplus* in Sass & Schmitt, 2010; Mx in Dolan et al., 2009; R in Reise et al., 2011), and likely will continue to be used, in simulation studies with EFA. Software- and study-specific postrotation algorithms for the identification and remediation of an equivalent form of $\boldsymbol{\eta}$ might have worked, and might continue to work well in many instances. Even in such instances, however, because the REREFACt package provides a general algorithm within freely available software, R, it can be used to verify the efficacy of the study- and software-specific algorithm implemented irrespective of the software program within which the simulation study occurred.

The ability to broadly generalize results from this study is constrained by the conditions imposed. Specific conditions that are restrictive include the exclusion of ESEM models with categorical and censored observed endogenous variables, multiple-group ESEM, and multiple-block EFA. Categorical and censored observed endogenous variables can be accommodated in ESEM under the general structural equation model proposed by Muthén (1984). Although we believe that the general algorithm implemented in REREFACt should work under this latent response variable formulation, as the key change is to arrays in the measurement model unaffected by indeterminacies within $\boldsymbol{\eta}$, future research in this area might be worthwhile. Subsequent studies that examine how REREFACt might need to be modified with multiple-group ESEM could advance the literature

and could be guided by similar work with confirmatory factor analysis (Asparouhov & Muthén, 2014) using component loss functions (Jennrich, 2006). Subsequent studies that examine how REREFAC might need to be modified to accommodate multiple-block EFA could advance the literature.

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APPENDIX A
COMPLETE COMMANDS FOR REPLICATIONS 1–100 AND 4,701–4,800 FROM EXAMPLE 1

```

Install.packages("REREFACT")
library(REREFACT)
data(pop_L_efa)
pop_L_efa
data(sample_lambda_efa)
sample_lambda_efa
data(sample_psi_efa)
sample_psi_efa
ex_efa = rerefact(n.factor = 3, n.eta = 3, n.var = 10, pop_lambda = pop_L_efa, sample_lambda = sample_lambda_efa)
names(ex_efa)
ex_efa$n.perm
ex_efa$permutation
ex_efa$replication.permutation
ex_efa$summary.permutation
ex_efa$correct.permutation
P_efa <- read.csv("P.csv")
correct_lambda(P_data = P_efa, rep = 200, n.eta = 3, sample_lambda = sample_lambda_efa)
correct_psi(P_data = P_efa, rep = 200, n.eta = 3, sample_psi = sample_psi_efa)

```

APPENDIX B
COMPLETE COMMANDS FOR REPLICATIONS 1–100 AND 4,701–4,800 FROM EXAMPLE 2

```

Install.packages("REREFACT")
library(REREFACT)
data(pop_L_esem)
pop_L_esem
data(sample_lambda_esem)
sample_lambda_esem
data(sample_psi_esem)
sample_psi_esem
data(sample_alpha_esem)
sample_alpha_esem
data(sample_beta_esem)
sample_beta_esem
data(sample_gamma_esem)
sample_gamma_esem
ex_esem = rerefact(3, 4, 10, pop_L_esem, sample_lambda_esem)
names(ex_esem)
ex_esem$n.perm
ex_esem$permutation
ex_esem$replication.permutation
ex_esem$summary.permutation
ex_esem$correct.permutation
P_esem <- read.csv("P.csv")
correct_lambda(P_esem, 200, 4, sample_lambda_esem)
correct_psi(P_esem, 200, 4, sample_psi_esem)
correct_alpha(P_esem, 200, 4, sample_alpha_esem)
correct_beta(P_esem, 200, 4, sample_beta_esem)
correct_gamma(P_esem, 200, 4, sample_gamma_esem)

```